Duration : 144 minutes

EPFL

Linear Algebra Exam

Common part

Fall 2021

Questions

For the **multiple choice** questions, we give

- $+3$ points if your answer is correct,
- 0 points if you give no answer or more than one,
- -1 if your answer is incorrect.

For the **true/false** questions, we give

- $+1$ points if your answer is correct,
- 0 points if you give no answer or more than one,
- -1 points if your answer is incorrect.

The notation and terminology of this exam are those used in the exercise sheets and the lectures of the course Linear Algebra given during the Fall semester 2021.

Notation (all standard)

– $\mathbb{R}$ denotes the set of real numbers.
– For a matrix A, $a_{ij} \in \mathbb{R}$ denotes the entry of A in row i and column j.
– For a vector $\boldsymbol{x} \in \mathbb{R}^n$, x_i denotes the ith coordinate of $\boldsymbol{x}$.
– I_m denotes the $m \times m$ identity matrix.
– $\mathbb{P}_n$ is the vector space of polynomials of degree less than or equal to n.
– The scalar or inner product of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ is defined as $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^T \boldsymbol{y}$.

First part: Multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct response.

Question 1 : The straight line that best approximates (in the sense of least squares) the (x, y) point data $(-2, -4)$, $(0, 0)$, $(1, 4)$, $(2, 1)$, $(4, -1)$ is

☐ $y = \frac{1}{2} - \frac{1}{2}x$

☐ $y = \frac{1}{2} + \frac{1}{2}x$

☐ $y = -\frac{1}{2} + \frac{1}{2}x$

☐ $y = -\frac{1}{2} - \frac{1}{2}x$

Question 2 : The inverse $B = A^{-1}$ of the matrix

$$A = \begin{pmatrix} 1 & -1 & -2 \\ -3 & 4 & 7 \\ 4 & -2 & -7 \end{pmatrix}$$

is such that

☐ $b_{32} = -2$ ☐ $b_{32} = 2$ ☐ $b_{32} = 1$ ☐ $b_{32} = -1$

Question 3 : Let A be a symmetric matrix such that

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

are three eigenvectors of A associated with, respectively, the three eigenvalues -5, 5, and 2. Then

☐ $A = \begin{pmatrix} 5 & -4 & 0 \\ -4 & -5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

☐ $A = \begin{pmatrix} 3 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & -3 \end{pmatrix}$

☐ $A = \begin{pmatrix} 3 & -4 & 0 \\ -4 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

☐ $A = \begin{pmatrix} 5 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & -5 \end{pmatrix}$

Question 4 : For $a \in \mathbb{R}$, the system

$$\begin{cases} x + ay + 3z = 0 \\ y - 2z = 3 \\ -x + 4y + 4z = a \\ (a+6)y + 3z = a^2 \end{cases}$$

☐ has at least one solution if and only if $a \in \{-2, 3\}$
☐ has infinitely many solutions if and only if $a = -2$
☐ has no solution if and only if $a \in \{-2, 3\}$
☐ has a unique solution if and only if $a = 3$

Question 5 : Let

$$A = \begin{pmatrix} 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 1 & 5 \\ 1 & 3 & 1 & 2 & 2 \\ 0 & 0 & 5 & 0 & 4 \end{pmatrix}.$$

Then

☐ $\det(A) = 6$ ☐ $\det(A) = -24$ ☐ $\det(A) = 0$ ☐ $\det(A) = 48$

Question 6 : Let R be the reduced echelon form of the matrix

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 1 & 5 & 2 & 5 \\ 3 & 3 & -6 & -2 \end{pmatrix}.$$

Then

☐ $r_{13} = -4$ ☐ $r_{13} = 0$ ☐ $r_{13} = -2$ ☐ $r_{13} = -3$

Question 7 : Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation with matrix

$$A = \begin{pmatrix} 0 & 5 & 4 & 4 \\ 2 & 1 & 1 & 5 \\ 4 & 3 & -5 & 3 \end{pmatrix},$$

(with respect to the standard bases for $\mathbb{R}^4$ and $\mathbb{R}^3$). Then

☐ T is bijective, i.e. both injective and surjective
☐ T is injective but not surjective
☐ T is surjective but not injective
☐ T is neither injective nor surjective

Question 8 : Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^4$ and $\mathcal{B} = \{1 + t^2, 2 - t^3, t, 1 - t^2\}$ a basis of $\mathbb{P}_3$.
Let $T : \mathbb{P}_3 \to \mathbb{R}^4$ be the linear transformation

$$T(a + bt + ct^2 + dt^3) = \begin{pmatrix} a - b \\ a - c \\ 2a + c \\ 2b + d \end{pmatrix}.$$

The matrix A for T with respect to the bases $\mathcal{B}$ of $\mathbb{P}_3$ and $\mathcal{E}$ of $\mathbb{R}^4$, i.e. $[T(p)]_{\mathcal{E}} = A[p]_{\mathcal{B}}$ for all $p \in \mathbb{P}_3$ is such that

☐ $a_{34} = -1$ ☐ $a_{34} = 2$ ☐ $a_{34} = -2$ ☐ $a_{34} = 1$

Question 9 : Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right\}$$

be two bases of $\mathbb{R}^3$. Then the change of coordinates matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ from the basis $\mathcal{B}$ to the basis $\mathcal{C}$, i.e. the matrix such that $[\boldsymbol{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\boldsymbol{x}]_{\mathcal{B}}$ for all $\boldsymbol{x} \in \mathbb{R}^3$, is

☐ $\begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 1 \end{pmatrix}$

☐ $\begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ -1 & -3 & 5 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & -3 \\ -3 & 1 & 5 \end{pmatrix}$

Question 10 : Given

$$A = \begin{pmatrix} 4 & 1 & -8 \\ 1 & 1 & 2 \\ 2 & 1 & -2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix},$$

the solution $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ of $A\boldsymbol{x} = \boldsymbol{b}$ has first coordinate

☐ $x_1 = -1$ ☐ $x_1 = 5$ ☐ $x_1 = 4$ ☐ $x_1 = -2$

Question 11 : Let

$$A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 2 & 1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} 5 \\ 20 \\ 0 \end{pmatrix}.$$

Then the least squares solution $\widehat{x} = \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix}$ of $A\boldsymbol{x} = \boldsymbol{b}$ satisfies

☐ $\widehat{x}_1 = -3$ ☐ $\widehat{x}_1 = -6$ ☐ $\widehat{x}_1 = 6$ ☐ $\widehat{x}_1 = 3$

Question 12 : Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The real roots of the characteristic polynomial of A are

☐ $\{0\}$ ☐ $\{-1, 0\}$ ☐ $\{0, 1\}$ ☐ $\{-1, 1\}$

Question 13 : The Gram-Schmidt algorithm applied to the columns of the matrix

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 1 & -1 & 6 \\ 1 & 1 & 3 \\ 0 & 1 & -6 \end{pmatrix}$$

yields an orthogonal basis of $\mathrm{Col}(A)$ given by the vectors

☐ $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ -6 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ -4 \end{pmatrix}$

☐ $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ -4 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \\ -6 \end{pmatrix}$

Question 14 : The orthogonal projection of the vector $\begin{pmatrix} 3 \\ 1 \\ -1 \\ 5 \end{pmatrix}$ on the subspace generated by the first two columns of the matrix A from Question 13 is the vector:

☐ $\begin{pmatrix} 21 \\ -15 \\ 3 \\ 9 \end{pmatrix}$ ☐ $\begin{pmatrix} 39 \\ -9 \\ 15 \\ 12 \end{pmatrix}$ ☐ $\begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 3 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

Question 15 : The matrix A from Question 13 has a QR-decomposition such that

☐ $r_{33} = 3\sqrt{2}$ ☐ $r_{33} = 2\sqrt{3}$ ☐ $r_{33} = \sqrt{3}$ ☐ $r_{33} = \sqrt{2}$

Question 16 : Let

$$\boldsymbol{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ -5 \end{pmatrix}, \qquad \boldsymbol{w}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 5 \end{pmatrix},$$

and $W = \operatorname{Span}\{\boldsymbol{w}_1, \boldsymbol{w}_2\} \subset \mathbb{R}^4$. Then a basis of the orthogonal complement $W^{\perp}$ is given by the vectors

- ☐ $\begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$
- ☐ $\begin{pmatrix} -6 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$
- ☐ $\begin{pmatrix} -6 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$
- ☐ $\begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 17 : If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is a linearly independent set of vectors in a vector space V and if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are any real numbers, then $\{\alpha_1 \boldsymbol{v}_1, \alpha_2 \boldsymbol{v}_2, \ldots, \alpha_k \boldsymbol{v}_k\}$ is also a set of linearly independent vectors in V.

☐ TRUE ☐ FALSE

Question 18 : The polynomials

$$p_1(t) = t^2\,, \quad p_2(t) = t^2 + t^3\,, \quad p_3(t) = t - t^3\,, \quad p_4(t) = 1 + t^3$$

form a basis for $\mathbb{P}_3$.

☐ TRUE ☐ FALSE

Question 19 : If A is a 3×3 matrix with eigenvalues 1, 2, and 4, then $\det(A) = 8$.

☐ TRUE ☐ FALSE

Question 20 : If A and B are two $n\times n$ invertible matrices, then AB is invertible and

$$(AB)^{-1} = A^{-1}B^{-1}\,.$$

☐ TRUE ☐ FALSE

Question 21 : The matrix

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

is diagonalizable.

☐ TRUE ☐ FALSE

Question 22 : The matrix

$$A = \begin{pmatrix} 10 & -11 & 11 \\ 11 & 11 & -11 \\ -11 & 11 & 10 \end{pmatrix}$$

is orthogonally diagonalizable.

☐ TRUE ☐ FALSE

Question 23 : Let A be a 6×6 matrix. If the characteristic polynomial of A is $p(\lambda) = (\lambda-3)^2\lambda^4$, then the geometric multiplicity of the eigenvalue $\lambda = 3$ is always 2.

☐ TRUE ☐ FALSE

Question 24 : Let V be a vector space such that $\dim V = n$. If S is a set of linearly independent vectors in V, then S is a basis for V.

☐ TRUE ☐ FALSE

Question 25 : Let A be a 9×5 matrix. If $\dim\big(\mathrm{Nul}(A)\big) = 5$, then $\dim\big(\mathrm{Col}(A)\big) = 0$.

☐ TRUE ☐ FALSE

Question 26 : Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 5 \\ 3 & 1 & 5 & -1 & -3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Given that the matrix R is the reduced echelon form for the matrix A, then the vectors

$$\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

form a basis for $\mathrm{Nul}(A)$.

☐ TRUE ☐ FALSE

Question 27 : Let A be the matrix from Question 26. Then the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

form a basis for $\mathrm{Col}(A)$.

☐ TRUE ☐ FALSE

Question 28 : Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors in $\mathbb{R}^n$. If $\|\boldsymbol{u} + \boldsymbol{v}\| = \|\boldsymbol{u} - \boldsymbol{v}\|$, then $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal.

☐ TRUE ☐ FALSE

Question 29 : Let $\mathcal{B}$ be the basis of $\mathbb{P}_2$ given by $\mathcal{B} = \{1,\ 1+t,\ 1+t^2\}$ and let $p \in \mathbb{P}_2$ be $p(t) = 3+4t+5t^2$. Then the first coordinate of p relative to the basis $\mathcal{B}$ is -6.

☐ TRUE ☐ FALSE

Question 30 : If A is an $n \times n$ matrix, then $\det(A^T) = -\det(A)$.

☐ TRUE ☐ FALSE

Question 31 : The set $\{p \in \mathbb{P}_4 \mid p(t) = at^4 \text{ for some } a \in \mathbb{R}\}$ is a subspace of $\mathbb{P}_4$.

☐ TRUE ☐ FALSE

Question 32 : Let

$$\boldsymbol{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

Then $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ is the closest point in $W = \mathrm{Span}\{\boldsymbol{u}, \boldsymbol{v}\} \subset \mathbb{R}^3$ to $\boldsymbol{y}$.

☐ TRUE ☐ FALSE

Question 33 : Let V and W be two finite-dimensional vector spaces and $T : V \to W$ a linear transformation. If $\dim W < \dim V$, then T is not injective.

☐ TRUE ☐ FALSE